

A classification of Newton polygons of L -functions on polynomials*

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Abstract

Considering the L -function of exponential sums associated to a polynomial over a finite field \mathbb{F}_q , Deligne proved that a reciprocal root's p -adic order is a rational number in the interval $[0, 1]$. Based on hypergeometric theory, in this paper we improve this result that there are only finitely many possible forms of Newton polygons for the L -function of degree d polynomials independent of p , when p is larger than a constant D^* (Theorem 4.3), i.e., a reciprocal root's p -adic order has form $\frac{up-v}{D^*(p-1)}$ in which u, v have finitely many possible values. Furthermore, when $p > D^*$, to determine the Newton polygon is only to determine it for any two specified primes $p_1, p_2 > D^*$ in the same residue class of D^* (Theorem 4.5).

1 Introduction

Let \mathbb{F}_q ($q = p^m$) be the finite field of q elements with characteristic p and \mathbb{F}_{q^r} be the extension of \mathbb{F}_q of degree r . Let ζ_p be a fixed primitive p -th root of unity in the complex numbers. For any Laurent polynomial $f(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, we form the exponential sum

$$S_r(f) = \sum_{x_i \in \mathbb{F}_{q^r}} \zeta_p^{\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(f(x_1, \dots, x_n))}.$$

The L -function is defined by

$$L(f, T) = \exp\left(\sum_{r=1}^{\infty} S_r(f) \frac{T^r}{r}\right).$$

Consider the case $n = 1$, and f is a polynomial with degree $d < p$. It follows from Weil's work on the Riemann hypothesis for function fields with characteristic p that this L -function is actually a polynomial of degree $d - 1$. We can write it as

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$$L(f, T) = (1 - \omega_1 T) \cdots (1 - \omega_{d-1} T).$$

Another work on it of Weil is that the reciprocal roots $\omega_1, \dots, \omega_{d-1}$ are q -Weil numbers of weight 1, i.e., algebraic integers, all of whose conjugates have complex absolute $q^{\frac{1}{2}}$. Moreover, for any prime $l \neq p$, they are l -adic units, that is, $|\omega_i|_l = 1$, while $|\omega_i|_p = q^{r_i}$ with $r_i \in \mathbb{Q} \cap [0, 1]$. Deligne proved in general that $|\omega_i| = q^{\frac{u_i}{2}}$ with $u_i \in \mathbb{Z} \cap [0, 2n]$, and $|\omega_i|_p = q^{-r_i}$ with $r_i \in \mathbb{Q} \cap [0, n]$ ([3] and [10]).

A natural question is how to determine their q -adic absolute value, or equivalently to determine r_i . In other words, one would like to determine the Newton polygon $NP_q(f)$ of $L(f, T)$ where NP_q means the Newton polygon taken with respect to the valuation \mathbf{ord}_q normalized by $\mathbf{ord}_{q^d} = 1$ (cf. [4], Chapter IV for the link between the Newton polygon of a polynomial and the valuations of its roots). There is an elegant general answer to this problem when $p \equiv 1 \pmod{d}$, $p \geq 5$: the Newton polygon $NP_q(f)$ has vertices (cf. [5] Theorem 7.5)

$$(n, \frac{n(n+1)}{2d})_{1 \leq n \leq d-1}$$

This polygon is often called the *Hodge polygon* for polynomials of degree d , and denoted by $HP(d)$. We know that $NP_q(f)$ lies above $HP(d)$.

Unfortunately, for general prime p , there is no such exact answer of $NP_q(f)$. $NP_q(f)$ has been only determined for polynomials with degree 3, 4, 6 ([2], [6] and [7] respectively), in which the case of degree 6 has some essential mistakes pointed by us in [13], via comparing with our results. In [13], we gave another method to calculate $NP_q(f)$ for polynomials with degree 5, 6, and the first $\lceil \sqrt{2d} \rceil + 2$ slopes in general for $q = p$.

Another result on $NP_q(f)$ is Hui Zhu's result ([8] and [9]), concerning the one-dimensional case of Wan's conjecture [10] as follows. Let the polygon $NP_q(f)$ be the Newton polygon obtained from the reduction of f modulo a prime above p in the field defined by the coefficients of f . Then there is a Zariski dense open subset \mathcal{U} defined over \mathbb{Q} in the space of polynomial of degree d such that, for every f in $\mathcal{U}(\bar{\mathbb{Q}})$, we have $\lim_{p \rightarrow \infty} NP_q(f) = HP(d)$.

The Grothendieck specialization theorem implies that, in the case of dimension one, for $p > d$, the lowest Newton polygon

$$GNP(d, p) = \inf NP_q(f)$$

exists when f runs over polynomials of degree d over \mathbb{F}_q , which is called *generic Newton polygon*.

Blache and Férard determined explicitly both the generic polygon $GNP(d, p)$ and the associated *Hasse polynomial* $H_{d,p}$ for $p \geq 3d$ [1].

Because of Deligne's work, when expanding $L(f, T) = \det(\mathbf{I}_{d-1} - T\Gamma^{\tau^{m-1}} \cdots \Gamma)$ (cf. [1], Proposition 1.1) directly, we can only consider the items whose p -adic orders are smaller than a fixed number. A fact that this partial algebraic sum can be expressed as a finite linear combination of hypergeometric polynomials will be proved in section 3 (Lemma 3.2). Combining with H.S.Wilf and D.Zeilberger's work [12], we improved Deligne's work that there are only finitely many possible forms of Newton polygons independent of p , when p is larger than a constant D^* (Theorem 4.3).

Furthermore, to determine $NP_q(f)$ for $p > D^*$, we need only to determine it for any two fixed primes $p_1, p_2 > D^*$ in the same residue class of p modular D^* (Theorem 4.5).

We use p -adic cohomology of Dwork, Robba and others, especially Blache and Férard's detailed description in their paper [1].

The rest of the paper is organized as below: In Section 2.1 we introduce some preliminaries on hypergeometric and holonomic functions, which are needed for proving our theorems in Section 4. We also introduce some concepts and results in Section 2.2 as basis of our work. In Section 3 we expand $L(f, T)$ in detail. Based on Lemma 3.2, we will prove our main theorem in Section 4.

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2 Preliminary

2.1 Hypergeometric and holonomic functions

In this section, we introduce the so called Fundamental Corollary in theory of hypergeometric and holonomic functions needed for us [12].

Definition 2.1 A function $F(a_1, \dots, a_m)$ of m discrete variables is a hypergeometric term if for every a_i ,

$$\frac{E_{a_i} F}{F} = \frac{P_i}{Q_i}, i = 1, \dots, m.$$

where P_i, Q_i are all polynomials in the variables (a_1, \dots, a_m) , and E_a are the shift operators acting on functions $f(a, \mathbf{b})$ by changing a to $a + 1$, i.e.

$$E_a f(a, \mathbf{b}) = f(a + 1, \mathbf{b}).$$

Phrased otherwise, F is a solution of the system of linear recurrence equations

$$(Q_i E_{a_i} - P_i) F = 0, i = 1, \dots, m.$$

If the dimension of the space of solutions of that system is *finite*, the functions F are called *holonomic*.

In [12] it is shown how to check for holonomicity, and in particular it is proved that the following class of *proper – hypergeometric functions* are holonomic (we omit the continuous variables part here).

Definition 2.2 A term $F(a_1, \dots, a_m)$ of m discrete variables is *proper-hypergeometric*, if it has the form

$$P(a_1, \dots, a_m) \prod_{i=1}^I (e_0^{(i)} + \sum_{j=1}^m e_j^{(i)} a_j)!^{g_i}$$

where $P(a_1, \dots, a_m)$ is a polynomial and $e_j^{(i)}$ and g_i are integers.

We need two more concepts for leading to the Fundamental Corollary.

Definition 2.3 A function $F(a_1, \dots, a_m)$ vanishes at infinity if for every variable a_i ,

$$\lim_{|a_i| \rightarrow \infty} F(\mathbf{a}) = 0.$$

Definition 2.4 An integral-sum

$$g(\mathbf{n}) := \sum_{\mathbf{k}} F(\mathbf{n}, \mathbf{k})$$

is pointwise trivially evaluable, if for every specific specialization of the auxiliary variables (parameters) \mathbf{n} there is an algorithm that will evaluate it.

Corollary 2.5 (Fundamental Corollary) Let $F(n, \mathbf{k})$ be hypergeometric and holonomic (both hold if it is proper-hypergeometric) in the discrete variables n and \mathbf{k} , and vanishes at infinity for every fixed n , then

$$f(n) := \sum_{\mathbf{k}} F(n, \mathbf{k})$$

satisfies a linear recurrence equation with polynomial coefficients

$$P(N, n)f(n) = 0.$$

where N is the shift operators acting on functions by changing n to $n + 1$.

H.S.Wilf and D.Zeilberger gave a method to find such $P(N, n)$, and gave an effective upper bound for the N -degree of $P(N, n)$ [12]. More detailed, assume

$$F(n, \mathbf{k}) = R(n, \mathbf{k}) \frac{\prod_{s=1}^p (a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s)!}{\prod_{s=1}^q (u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s)!}$$

where $\mathbf{k} = (k_1, \dots, k_r)$, R is a polynomial, the a 's, u 's, \mathbf{b} 's and \mathbf{v} 's are integers that contain no additional parameters, and the c 's and w 's are integers that may involve unspecified parameters (in our case the prime p is such parameter in c 's and w 's).

Let

$$f_n(\mathbf{x}) = \sum_{\mathbf{k}} F(n, \mathbf{k}) \mathbf{x}^{\mathbf{k}}.$$

Then there exist polynomials $\alpha_j(n, \mathbf{x})$ independent of \mathbf{k} satisfying

$$\sum_{j=0}^J \alpha_j(n, \mathbf{x}) f_{n-j}(\mathbf{x}) = 0$$

where the coefficients

$$\alpha_j(n, \mathbf{x}) = \sum_{0 \leq i_1, \dots, i_r \leq I} \alpha_{\mathbf{i}, j}(n) \mathbf{x}^{\mathbf{i}}$$

and I, J are both bounded by $a, \mathbf{b}, u, \mathbf{v}, \deg(R)$.

2.2 p -Adic differential operators and exponential sums

Let $f(x) := \alpha_d x^d + \cdots + \alpha_1 x$, $\alpha_d \neq 0$, be a polynomial of degree $d < p$, over the field \mathbb{F}_q , and let $g(x) := a_d x^d + \cdots + a_1 x \in \mathcal{O}_m[x]$ be the polynomial whose coefficients are the Teichmüller lifts of those of f . Let $A := B(0, 1^+) \setminus B(0, 1^-)$. We consider the space $\mathcal{H}^\dagger(A)$ of overconvergent analytic functions on A . Define the function $H := \exp(\pi g(x))$ and let D be the differential operator

$$D := x \frac{d}{dx} + \pi x g'(x) \quad (= H^{-1} \circ x \frac{d}{dx} \circ H).$$

Then for every $n \in \mathbb{Z}$, x^n can be written uniquely as

$$x^n \equiv \sum_{i=0}^{d-1} a_{ni} x^i \pmod{D\mathcal{H}^\dagger(A)}$$

for some $a_{ni} \in \mathcal{K}_m(\pi)$, $0 \leq i \leq d-1$, where \mathcal{K}_m is an unramified extension of degree m of the p -adic numbers field \mathbb{Q}_p .

We define the power series $\theta(x) := \exp(\pi x - \pi x^p)$ and $F(x) := \prod_{i=1}^d \theta(a_i x^i) := \sum_{n \geq 0} h_n x^n$. Define a mapping ψ_q on $\mathcal{H}^\dagger(A)$ by $\psi_q f(x) := \frac{1}{q} \sum_{z^q=x} f(z)$. Let β be the endomorphism of $\mathcal{H}^\dagger(A)$ defined by $\beta = \psi_p \circ F$; then $\tau^{-1} \circ \beta$ (τ being the Frobenius) commutes with D up to a factor p , and passes to the quotient, giving an endomorphism $\tau^{-1} \circ \beta$ of W , the $\mathcal{K}_m(\zeta_p)$ -vector space with basis \mathcal{B} .

Let $M := \text{Mat}_{\mathcal{B}}(\beta)$ be the matrix of β in the basis \mathcal{B} , and m_{ij} ($0 \leq i, j \leq d-1$) be the coefficients of M . From the description of F , we can write $m_{ij} = h_{pi-j} + \sum_{n \geq d} h_{np-j} a_{ni}$ (cf. [11], 7.10). Define $\Gamma := (m_{ij})_{1 \leq i, j \leq d-1}$, then

$$L(f, T) = \det(\mathbf{I}_{d-1} - T\Gamma^{\tau^{m-1}} \cdots \Gamma)$$

(cf. [1], Proposition 1.1).

3 Finite sum expression of \mathcal{M}_n

Let $f(x) := \alpha_d x^d + \cdots + \alpha_1 x$, $\alpha_d \neq 0$, be a polynomial of degree $d < p$, over the field \mathbb{F}_q . Denote $L(f, T) = 1 + \sum_{n=1}^{d-1} \mathcal{M}_n T^n$, our aim is to determine every $\text{ord}_q \mathcal{M}_n$.

We expand $\det(\mathbf{I}_{d-1} - T\Gamma^{\tau^{m-1}} \cdots \Gamma)$, i.e.

$$\mathcal{M}_n = \sum_{1 \leq u_1 < \cdots < u_n \leq d-1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \left(\sum_{1 \leq k_1, \dots, k_{m-1} \leq d-1} m_{u_i, k_1}^{\tau^{m-1}} m_{k_1, k_2}^{\tau^{m-2}} \cdots m_{k_{m-1}, u_{\sigma(i)}} \right). \quad (1)$$

3.1 Finite sum expression of m_{ij}

Set $\theta(x) := \sum_{i \geq 0} b_i x^i$; then we have (Lemma 1.2 in [1])

- (i) $\text{ord}_p b_i \geq \frac{i}{p-1}$ for $0 \leq i \leq p^2 - 1$;
- (ii) $\text{ord}_p b_i \geq \frac{(p-1)i}{p^2}$ for $i \geq p^2$.

Since

$$h_n = \sum_{i_1 + \dots + di_d = n} a_1^{i_1} \dots a_d^{i_d} b_{i_1} \dots b_{i_d},$$

then

Lemma 3.1 *We have the relations*

- (i) $\text{ord}_p h_i \geq \frac{i}{d(p-1)}$ for $0 \leq i \leq p^2 - 1$;
- (ii) $\text{ord}_p h_i \geq \frac{(p-1)i}{dp^2}$ for $i \geq p^2$.

Proof: For $0 \leq i \leq p^2 - 1$ and $i_1 + \dots + di_d = i$, we have $0 \leq i_1, \dots, i_d \leq p^2 - 1$ and

$$\text{ord}_p b_{i_1} \dots b_{i_d} = \text{ord}_p b_{i_1} + \dots + \text{ord}_p b_{i_d} \geq \frac{i_1}{p-1} + \dots + \frac{i_d}{p-1} \geq \frac{i}{d(p-1)},$$

then (i) holds.

For $i \geq p^2$ and $i_1 + \dots + di_d = i$, since $\frac{1}{p-1} \geq \frac{p-1}{p^2}$, we have

$$\text{ord}_p b_{i_1} \dots b_{i_d} = \text{ord}_p b_{i_1} + \dots + \text{ord}_p b_{i_d} \geq \frac{(p-1)i_1}{p^2} + \dots + \frac{(p-1)i_d}{p^2} \geq \frac{(p-1)i}{dp^2},$$

then (ii) holds. □

Furthermore,

$$h_n = \sum_{k \geq 0} \pi^k \sum_{\substack{\sum_{i=1}^d i(m_i + (p-1)n_i) = n, \\ \sum_{i=1}^d m_i = k}} \prod_{i=1}^d \frac{(-1)^{n_i}}{m_i!} \binom{m_i}{n_i} a_i^{m_i + (p-1)n_i},$$

following the definition of m_{ij} in Section 2.2, we have

$$m_{ij} = \sum_{r > 0} h_{rp-j} a_{ri} = \sum_{r > 0} a_{ri} \sum_{k \geq 0} \pi^k \sum_{\substack{\sum_{l=1}^d l(m_l + (p-1)n_l) = rp-j, \\ \sum_{l=1}^d m_l = k}} \prod_{l=1}^d \frac{(-1)^{n_l}}{m_l!} \binom{m_l}{n_l} a_l^{m_l + (p-1)n_l}. \quad (2)$$

Since the p -adic order of the reciprocal roots of $L(f, T)$ are all smaller than 1, we can consider only the items in (1) whose p -adic orders are smaller than n . If $p > 3$, then $\frac{(p-1)(rp-j)}{dp^2} - \frac{r-i}{d(p-1)} > n$ when $r \geq 3nd$ and $1 \leq i, j \leq d-1$. Following Lemma 1.1 in [1], and equation (1) and Lemma 3.1, we should consider only the part $r < 3nd$ in the sum of equation (2).

If $p \geq 3d^2$, then any n_l in equation (2) must smaller than $3nd$ when $r < 3nd$. We can rewrite m_{ij} to

$$m_{ij} = \sum_{r \geq 3nd} h_{rp-j} a_{ri} +$$

$$\sum_{0 < r < 3nd} a_{ri} \sum_{0 \leq n_1, \dots, n_d < 3nd} \sum_{k \geq 0} \pi^k \sum_{\substack{\sum_{l=1}^d l(m_l + (p-1)n_l) = rp-j, \\ \sum_{l=1}^d m_l = k}} \prod_{l=1}^d \frac{(-1)^{n_l}}{m_l!} \binom{m_l}{n_l} a_l^{m_l + (p-1)n_l}. \quad (3)$$

3.2 Finite sum expression of \mathcal{M}_n

Let

$$s_l = k - \sum_{t=1}^l m_t$$

for $1 \leq l \leq d-1$ and $s_0 = k$ and $s_d = 0$. Then

$$m_l = s_{l-1} - s_l$$

for $1 \leq l \leq d$.

We transform (3) into

$$\sum_{0 < r < 3nd} a_{ri} \sum_{0 \leq n_1, \dots, n_d < 3nd} \sum_{k \geq 0} \pi^k \sum_{\substack{\sum_{l=1}^d l(s_{l-1} - s_l + (p-1)n_l) = rp-j}} \prod_{l=1}^d \frac{(-1)^{n_l}}{(s_{l-1} - s_l)!} \binom{s_{l-1} - s_l}{n_l} a_l^{s_{l-1} - s_l + (p-1)n_l}.$$

Note that the sum $\sum_{l=1}^d l(s_{l-1} - s_l + (p-1)n_l) = rp-j$ is equivalent to

$$s_{d-1} = rp-j - \sum_{l=1}^{d-1} s_{l-1} + \sum_{l=1}^d l(p-1)n_l,$$

we can omit $\sum_{\sum_{l=1}^d l(s_{l-1} - s_l + (p-1)n_l) = rp-j}$ by substituting s_{d-1} into the expression.

Let

$$\omega = \{\sigma, u_i, k_j, r_{i,j}, n_{i,j,l}\}_{i=1, \dots, n; j=1, \dots, m; l=1, \dots, d}$$

where $0 \leq n_{i,j,l} < 3nd$.

Let $p_{i,j} = r_{i,j}p - k_j$ for $i = 1, \dots, n; j = 1, \dots, m-1$ and $p_{i,m} = r_{i,m}p - u_{\sigma(i)}$ for $i = 1, \dots, n$.

Denote $(\{s_{i,j,0} = k_{i,j}, s_{i,j,d-1} = p_{i,j} - \sum_{l=1}^{d-1} s_{l-1} + \sum_{l=1}^d l(p-1)n_{i,j,l} \text{ and } s_{i,j,d} = 0\})$

$$F_\omega(\{s_{i,j,0}, s_{i,j,1}, \dots, s_{i,j,d-2}\}) = \prod_{i=1, \dots, n; j=1, \dots, m} \prod_{l=1}^d \frac{(-1)^{n_{i,j,l}}}{(s_{i,j,l-1} - s_{i,j,l})!} \binom{s_{i,j,l-1} - s_{i,j,l}}{n_{i,j,l}} a_l^{s_{i,j,l-1} - s_{i,j,l} + (p-1)n_{i,j,l}}.$$

Then F_ω is hypergeometric and holonomic. Replacing $k_{n,m}$ with variable $0 \leq a$ and $0 < k \leq p-1$ such that $k_{n,m} = a(p-1) + k - \sum_{(i,j) \neq (n,m)} k_{i,j}$.

Furthermore, let

$$G_{\omega,a}(k) = \sum_{s_{i,j,0}, s_{i,j,1}, \dots, s_{i,j,d-1}} F_{\omega}(\{s_{i,j,0}, s_{i,j,1}, \dots, s_{i,j,d-2}\})$$

Then we have

$$\begin{aligned} \mathcal{M}_n = & \sum_{1 \leq u_1 < \dots < u_n \leq d-1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{1 \leq k_1, \dots, k_{m-1} \leq d-1} \sum_{0 < r_{i,j} < 3nd} \\ & \prod_{1 \leq i \leq n} a_{r_{i,1}, u_i}^{\tau^{m-1}} \prod_{1 \leq i \leq n, 1 < j \leq m} a_{r_{i,j}, k_{j-1}}^{\tau^{m-j}} \left(\sum_{0 \leq n_{i,j,1}, \dots, n_{i,j,d} < 3nd} \sum_{0 \leq a} (-p)^a \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \right) + \mathcal{N}_n \end{aligned} \quad (4)$$

where the p -adic order of \mathcal{N}_n is not smaller than d .

Since $k_{i,j} \leq r_{i,j}p - 1 < 3ndp$, we have

$$a(p-1) + k = \sum_{i,j} k_{i,j} < 3dmn^2p,$$

thus $a \leq 3dmn^2$. Therefore we can consider only

$$\begin{aligned} \widetilde{\mathcal{M}}_n = & \sum_{1 \leq u_1 < \dots < u_n \leq d-1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{1 \leq k_1, \dots, k_{m-1} \leq d-1} \sum_{0 < r_{i,j} < 3nd} \\ & \prod_{1 \leq i \leq n} a_{r_{i,j}, u_i}^{\tau^{m-1}} \prod_{1 \leq i \leq n, 1 < j \leq m} a_{r_{i,j}, k_{j-1}}^{\tau^{m-j}} \left(\sum_{0 \leq n_{i,j,1}, \dots, n_{i,j,d} < 3nd} \sum_{0 \leq a \leq 3dmn^2} (-p)^a \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \right) \end{aligned} \quad (5)$$

instead of \mathcal{M}_n . We have

Lemma 3.2 $\text{ord}_p \mathcal{M}_n$ is congruent with $\frac{c}{p-1}$ or some $\frac{c}{p-1} + \text{ord}_p \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \pmod{1}$, where $-3mn^2 < c \leq 0$ is an integer independent of p .

Proof: By Lemma 1.1. in [1], $0 \geq \text{ord}_p a_{r,i} := \frac{s}{p-1} \geq -\frac{r-i}{d(p-1)}$. When we restrict r to $r < 3nd$, we have $0 \geq s > -3n$.

For any $r_{i,j}$, u_i and k_j , let $\text{ord}_p \prod_{1 \leq i \leq n} a_{r_{i,j}, u_i}^{\tau^{m-1}} \prod_{1 \leq i \leq n, 1 < j \leq m} a_{r_{i,j}, k_{j-1}}^{\tau^{m-j}} := \frac{z}{p-1}$. Then it is within the range

$$0 \geq \frac{z}{p-1} > -\frac{3n^2m}{p-1}.$$

Besides,

$$\begin{aligned} & \sum_{1 \leq u_1 < \dots < u_n \leq d-1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{1 \leq k_1, \dots, k_{m-1} \leq d-1} \\ & \sum_{0 < r_{i,j} < 3nd} \prod_{1 \leq i \leq n} a_{r_{i,j}, u_i}^{\tau^{m-1}} \prod_{1 \leq i \leq n, 1 < j \leq m} a_{r_{i,j}, k_{j-1}}^{\tau^{m-j}} \left(\sum_{0 \leq n_{i,j,1}, \dots, n_{i,j,d} < 3nd} \sum_{0 \leq a \leq 3dmn^2} (-p)^a \right) \end{aligned} \quad (6)$$

is a finite sum independent of p . In other words, $\text{ord}_p \mathcal{M}_n$ is congruent with the p -adic order of one of the finitely many numbers items in the sum of (6) $\pmod{1}$.

The result follows. \square

4 Finite possible forms of Newton polygon

Since F_ω is hypergeometric and holonomic, following the Fundamental Corollary in holonomic theory, there exist a non-zero operator $P_{\omega,a}(K, k)$ (a polynomial of k and shift operator K defined by $K \circ f(k) = f(k+1)$) annihilate $G_{\omega,a}(k)$, i.e.

$$P_{\omega,a}(K, k)G_{\omega,a}(k) \equiv 0.$$

By the definition of $G_{\omega,a}(k)$, for any prime p we have

$$\text{ord}_p G_{\omega,a}(k) \equiv 0 \pmod{1}$$

Suppose $P_{\omega,a}(K, k) = \sum_{i=0}^{e-1} K^i P_i(k) \in \mathbb{Z}[K, k]$, where e is independent of p (see the end of Section 2.1). Let P_{i_0}, \dots, P_{i_h} be all of such P_i satisfying

$$p \nmid P_{i_j}$$

where $j = 0, \dots, r$ and $e-1 \geq i_0 > \dots > i_r \geq 0$. (If each of P_i is divisible by p , we can divide by p to reduce $P_{\omega,a}(K, k)$ until such P_{i_j} appears.)

Proposition 4.1 *If*

$$\text{ord}_p \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \equiv \frac{k_{\omega,a}}{p-1} \pmod{1}$$

where $e < k_{\omega,a} \leq p - e$, then

$$P_{i_0}(k_{\omega,a}) \equiv 0 \pmod{p}.$$

Proof:

Considering the equation

$$P_{\omega,a}(K, k_{\omega,a} - i_0)G_{\omega,a}(k_{\omega,a} - i_0) = 0$$

i.e.,

$$\sum_{i=0}^{e-1} K^i P_i(k_{\omega,a} - i_0)G_{\omega,a}(k_{\omega,a} - i_0) = \sum_{i=0}^{e-1} P_i(k_{\omega,a} - i_0 + i)G_{\omega,a}(k_{\omega,a} - i_0 + i) = 0. \quad (7)$$

We will show that

$$\text{ord}_p G_{\omega,a}(k_{\omega,a}) < \text{ord}_p P_i(k_{\omega,a} - i_0 + i)G_{\omega,a}(k_{\omega,a} - i_0 + i) \quad (8)$$

for all $0 \leq i \leq e-1$.

Since

$$\text{ord}_p G_{\omega,a}(k) \equiv 0 \pmod{1},$$

the congruences of $\text{ord}_p \pi^k G_{\omega,a}(k) \pmod{1}$ for $k = 1, \dots, p-1$ are all different.

Furthermore, by the definition,

$$\mathbf{ord}_p \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \equiv \frac{k_{\omega,a}}{p-1} (\mathbf{mod} \ 1),$$

then for any $k \neq k_{\omega,a}$, $\mathbf{ord}_p \pi^{k_{\omega,a}} G_{\omega,a}(k_{\omega,a}) < \mathbf{ord}_p \pi^k G_{\omega,a}(k)$.

For any $0 < k < k_{\omega,a}$, from the discussion above, we have

$$\begin{aligned} \frac{k_{\omega,a}}{p-1} + \mathbf{ord}_p G_{\omega,a}(k_{\omega,a}) &= \mathbf{ord}_p \pi^{k_{\omega,a}} G_{\omega,a}(k_{\omega,a}) < \mathbf{ord}_p \pi^k G_{\omega,a}(k) = \frac{k}{p-1} + \mathbf{ord}_p G_{\omega,a}(k) \\ &< \frac{k_{\omega,a}}{p-1} + \mathbf{ord}_p G_{\omega,a}(k), \end{aligned}$$

thus

$$\mathbf{ord}_p G_{\omega,a}(k_{\omega,a}) < \mathbf{ord}_p G_{\omega,a}(k).$$

For any $k_{\omega,a} < k \leq p-1$, we also have

$$\begin{aligned} \mathbf{ord}_p G_{\omega,a}(k_{\omega,a}) &< \frac{k_{\omega,a}}{p-1} + \mathbf{ord}_p G_{\omega,a}(k_{\omega,a}) = \mathbf{ord}_p \pi^{k_{\omega,a}} G_{\omega,a}(k_{\omega,a}) < \mathbf{ord}_p \pi^k G_{\omega,a}(k) = \frac{k}{p-1} + \mathbf{ord}_p G_{\omega,a}(k) \\ &\leq 1 + \mathbf{ord}_p G_{\omega,a}(k), \end{aligned}$$

i.e.,

$$\mathbf{ord}_p G_{\omega,a}(k_{\omega,a}) < 1 + \mathbf{ord}_p G_{\omega,a}(k).$$

Recall the definition of i_0 ,

$$p \mid P_i$$

for all $i > i_0$, then (8) holds. Combining with (7), we have

$$\mathbf{ord}_p P_{i_0}(k_{\omega,a}) > 0.$$

It shows that either $P_{i_0}(k_{\omega,a}) = 0$ or $\mathbf{ord}_p P_{i_0}(k_{\omega,a}) \geq 1$, i.e.,

$$P_{i_0}(k_{\omega,a}) \equiv 0 (\mathbf{mod} \ p).$$

□

Since prime p is arbitrary, following Proposition 4.1 we have

Corollary 4.2 *If $e < k_{\omega,a} \leq p - e$, then $k_{\omega,a}|_{p=0}$ is a rational root of $P_{i_0}|_{p=0}$, i.e.,*

$$P_{i_0}|_{p=0}(k_{\omega,a}|_{p=0}) = 0$$

If $0 < k_{\omega,a} \leq e$ or $p - e < k_{\omega,a} \leq p - 1$ for some fixed index ω , by the definition of $k_{\omega,a}$, i.e., $\mathbf{ord}_p \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) \equiv \frac{k_{\omega,a}}{p-1} (\mathbf{mod} \ 1)$, the congruence

$$\frac{c}{p-1} + \mathbf{ord}_p \sum_{0 < k \leq p-1} \pi^k G_{\omega,a}(k) (\mathbf{mod} \ 1)$$

in Lemma 3.2 with range $(\frac{-3mn^2-e+1}{p-1}, \frac{e+1}{p-1})$ is independent of p in the finite sum (6). Combining with Corollary 4.2, we have

Theorem 4.3 Let D^* be the least common multiple of all denominators of rational roots of such $P_{i_0}|_{p=0}$. Suppose $p > D^*$ and p in a fixed residue class of D^* . Denote p^* the inverse of $p \pmod{D^*}$. Thus

$$\lim_{p \rightarrow \infty} \mathbf{ord}_p \mathcal{M}_n = \frac{-p^* r}{D^*} \pmod{1}$$

where $r = 0$ or $\frac{r}{D^*}$ is congruent with a rational root of some $P_{i_0}|_{p=0} \pmod{1}$

Proof: Note that the sum of (6) has finitely many numbers of items, by Lemma 3.2, $\mathbf{ord}_p \mathcal{M}_n$ must be congruent with one of

$$\frac{k_{\omega,a} + c}{p-1} \pmod{1}$$

in which c is an integer with $-3mn^2 < c \leq 0$.

If $k_{\omega,a} \leq e$ or $k_{\omega,a} > p - e$, when $p \rightarrow \infty$ the limit of $\frac{k_{\omega,a} + c}{p-1} \pmod{1}$ is equal to 0.

Following Corollary 4.2, if $e < k_{\omega,a} \leq p - e$, then $P_{i_0}|_{p=0}(k_{\omega,a}|_{p=0}) = 0$, i.e. $k_{\omega,a}|_{p=0}$ is a rational root of $P_{i_0}|_{p=0}$.

Suppose $k_{\omega,a}|_{p=0} = \frac{r}{D^*}$, then $k_{\omega,a}$ has the form

$$k_{\omega,a} = \frac{sp + r}{D^*}.$$

Note that p is prime to D^* and $k_{\omega,a}$ is an integer, i.e., $sp + r \equiv 0 \pmod{D^*}$, then $s \equiv -p^* r \pmod{D^*}$.

So $\lim_{p \rightarrow \infty} \frac{k_{\omega,a}}{p-1} = \frac{s}{D^*} = \frac{-p^* r}{D^*} \pmod{1}$ is determined only by $\frac{r}{D^*}$, the rational root of $P_{i_0}|_{p=0}$. In this case

$$\lim_{p \rightarrow \infty} \mathbf{ord}_p \mathcal{M}_n = \frac{-p^* r}{D^*} \pmod{1}.$$

The theorem is proved. \square

Remark 4.4 Combining with Lemma 3.2, this theorem shows that the Newton polygon of $L(f, T)$ has finitely many possible forms when $p > D^*$. This means, we can classify the Newton polygons according to their limits when $p \rightarrow \infty$. In detail, the finite sum of (6) has

$$t = \binom{d-1}{n} n! (d-1)^{m-1} (3nd-1)^{n(m-1)} (3nd)^{dn(m-1)} (3dmn^2 + 1)$$

items, and then $\mathbf{ord}_p \mathcal{M}_n$ has at most t numbers of possible values when $p \rightarrow \infty$.

Suppose $p > D^*$ in a fixed residue class of D^* and $\mathbf{ord}_p \mathcal{M}_n = \frac{up-v}{D^*(p-1)}$, i.e.,

$$\mathbf{ord}_p \mathcal{M}_n = \frac{u}{D^*} \cdot \frac{p}{p-1} - \frac{v}{D^*} \cdot \frac{1}{p-1}. \quad (9)$$

If we know the values of $\mathbf{ord}_p \mathcal{M}_n$ on two specified primes $p_1, p_2 > D^*$ which are in the same residue class of D^* with p , saying $\mathbf{ord}_{p_1} \mathcal{M}_n = r_1$, $\mathbf{ord}_{p_2} \mathcal{M}_n = r_2$, then

$$\frac{u}{D^*} = \frac{r_1(p_1-1) - r_2(p_2-1)}{p_1 - p_2}$$

is determined. Thus

$$\frac{v}{D^*} = \left(\frac{u}{D^*} \frac{p_1}{p_1 - 1} - r_1 \right) (p_1 - 1)$$

is also determined. So $\text{ord}_p \mathcal{M}_n$ is determined in general by (9). i.e., we have

Theorem 4.5 *Let $p > D^*$. To determine the Newton polygon of $L(f, T)$, we need only calculate it on two specified values of prime p in each residue class of D^* .*

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